ON THE INVARIANT SUBSPACE PROBLEM FOR DISSIPATIVE OPERATORS IN KREIN SPACES

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Abstract. We relax assumptions for a dissipative operator in Krein space to possess a maximal non-negative invariant subspace. Our main result is a generalization of a well-known Pontrjagin-Krein-Langer-Azizov theorem. Then we investigate the semigroup properties of the restriction of the operator onto the invariant subspace.

KEY WORDS: Pontrjagin space, Krein space, invariant subspace problem.

Let H be a separable Hilbert space and $J = P_+ - P_-$ be a canonical symmetry ($J^2 = P_+ + P_- = 1$). The space H equipped with indefinite inner product

$$[x, y] = (Jx, y), \qquad x, y \in H$$

is called Krein space and denoted by $\mathcal{K} = \{H, J\}$ (or Pontrjagin space $\Pi_\varkappa = \{H, J\}$ if $\operatorname{rank} P_+ = \varkappa < \infty$).

Definitions. 1. A subspace $\mathcal{L}$ is *nonnegative* in $\mathcal{K}$ if $[x, x] \geqslant 0$ $\forall x \in \mathcal{L}$. It is maximal nonnegative if there are no proper extensions of $\mathcal{L}$.

2. An operator A is *dissipative* in H if

$$\operatorname{Im}(Ax, x) \geqslant 0 \qquad \forall x \in \mathcal{D}(A).$$

It is maximal dissipative if there are no proper dissipative extensions of A ($\Leftrightarrow \mathbb{C}^- \subset \rho(A)$, where $\mathbb{C}^-$ is the open lower-half plane).

3. A is dissipative in Krein space $\mathcal{K} = \{H, J\}$ if JA is dissipative in H. A is m-dissipative in $\mathcal{K}$ if JA is m-dissipative in H.

Symmetric and self-adjoint operators in $\mathcal{K}$ are defined analogously.

Let $H = H_+ \oplus H_-$, $H_\pm = P_\pm(H)$, $\mathcal{D}_\pm = \mathcal{D}(A) \cap H_\pm$.

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Assumption. *$\mathcal{D}(A) = \mathcal{D}_+ \oplus \mathcal{D}_-$ (actually, it is sufficient to assume that $\mathcal{D}_+ \oplus \mathcal{D}_-$ is a core of A) $\Leftrightarrow$ A admits matrix representation*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} P_+AP_+ & P_+AP_- \\ P_-AP_+ & P_-AP_- \end{pmatrix},$$

where $x = x_+ + x_-$ are identified with the columns $x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix}$.

Background

Theorem (Sobolev, 1941, 1962)**.** *A selfadjoint operator in Π_1 has at least one eigenvector corresponding to an eigenvalue $\lambda \in \overline{\mathbb{C}}^+$.*

Theorem (Pontrjagin, 1944)**.** *Let A be selfadjoint in $\Pi_\varkappa$, $\varkappa < \infty$. Then*

(a) *$\exists$ maximal nonnegative subspace $\mathcal{L}^+$ invariant with respect to A;*
(b) *among these subspaces $\exists \mathcal{L}^+$ such that $\sigma(A^+) \subset \overline{\mathbb{C}}^+$, $A^+ = A|_{\mathcal{L}^+}$.*

Theorem (Langer, 1961)**.** *Let A be selfadjoint in $\mathcal{K}$ and*

(i) *$\mathcal{D}(A) \supset H_+$ ($\Leftrightarrow$ A_{11} and A_{21} are bounded);*
(ii) *A_{12} is compact.*

Then the properties (a) and (b) hold.

Theorem (Krein, 1948, 1964)**.** *Analogues of Pontrjagin and Langer theorems are true for unitary operators in $\Pi_\varkappa$ and $\mathcal{K}$, respectively.*

M. Krein proposed a shorther elegant approach to prove the property (a) by means of the Shauder–Tikhonov fixed point theorem.

Theorem (Krein and Langer, 1971; Azizov, 1972)**.** *Let A be m-dissipative in $\Pi_\varkappa$. Then (a) and (b) hold.*

Theorem (Azizov, Khoroshavin, 1981)**.** *Let A be a contraction in Krein space and A_{12} be compact. Then (a) and (b) hold if $\mathbb{C}^-$ is replaced by the open unit disk.*

Theorem (Azizov, 1985)**.** *The analogue of the previous result holds for m-dissipative operators in $\mathcal{K}$ provided that $\mathcal{D}(A) \supset H_+$ and A_{12} is A_{22}-compact.*

Theorem (Shkalikov, 2004)**.** *Let*

(i) *A be dissipative in $\mathcal{K}$;*
(ii) *A_{22} be m-dissipative in H_- ($\Leftrightarrow \exists (A_{22} - \mu)^{-1}$ for some $\mu \in \mathbb{C}^-$);*
(iii) *$F(\mu) := (A_{22} - \mu)^{-1} A_{21}$ be bounded;*
(iv) *$G(\mu) := A_{12}(A_{22} - \mu)^{-1}$ be compact;*
(v) *$S(\mu) := A_{11} - A_{12}(A_{22} - \mu)^{-1} A_{21}$ be bounded.*

Then (a) and (b) hold.

The main novelty of this theorem is that the Langer condition $\mathcal{D}(A) \supset H_+$ is dropped out. In particular, for a model matrix operator

$$A = \begin{pmatrix} u(x) & \dfrac{d}{dx} \\ \dfrac{d}{dx} & \dfrac{d^2}{dx^2} \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is selfadjoint in $\mathcal{K} = \{H, J\}$, $H = L_2[0,1] \times L_2[0,1]$, provided that the domain of A is chosen properly, one can guarantee the validity of properties (a) and (b).

The main goal of this report is to prove the properties (a) and (b) provided that only assumptions (i)-(iv) are valid.

It turns out that we need no assumptions for the transfer function $S(\mu)$.

New problems arise if we start working with unbounded entries and reject Langer condition $\mathcal{D}(A) \supset H_+$. In this case, if we succeed to prove (a) and (b), we come to the following interesting problems

(c) does the operator $A^+ = A|_{\mathcal{L}^+}$ generate a C_0-semigroup, or holomorphic semigroup?

We shall provide some sufficient conditions for positive answer to this question.

A subspace $\mathcal{L}$ is A-invariant in classical sense if $\mathcal{L} \subset \mathcal{D}(A)$ and $A(\mathcal{L}) \subset \mathcal{L}$. We accept the following

Definition. $\mathcal{L}$ is A-invariant if $\mathcal{D}(A) \cap \mathcal{L}$ is dense in $\mathcal{L}$ and $Ax \in \mathcal{L}$ for all $x \in \mathcal{D}(A) \cap \mathcal{L}$.

Let us formulate the main results.

Theorem A. *Conditions (i)–(iv) imply property (a), i.e. there exists an A-invariant maximal non-negative subspace $\mathcal{L}^+$. The property $\mathcal{L}^+ \subset \mathcal{D}(A)$ (i.e. $\mathcal{L}^+$ is A^+-invariant in classical sense) holds if and only if $S(\mu)$ is bounded.*

Theorem B. *Property (b) holds if and only if assumption (i) is replaced by*

(i') *A is m-dissipative in $\mathcal{K}$.*

For short we accept the following notation.

Definition. We say that the operator B is a generator of H_0-semigroup if $\forall \varepsilon > 0$ the operator $B - \varepsilon$ generates a holomorphic semigroup.

Remind that A^+ is the restriction of A onto the invariant subspace $\mathcal{L}^+$.

Theorem C. *iA^+ generates a C_0-semigroup of exponential type 0 is one of the following conditions holds*

(1) *A_{12} is compact;*
(2) *$-iA_{22}$ generates an H_0-semigroup.*

Theorem D. *iA^+ generates an exponentially stable semigroup if either assumption (1) or (2) of Theorem C holds and A is uniformly dissipative in $\mathcal{K}$.*

Theorem E. *There is $\mu \in \mathbb{C}^+$ such that $iS(\mu)$ generates an H_0-semigroup. Then iA^+ generates H_0-semigroup.*

Proof of the first two theorems

Assumptions (ii)–(iv) allow to use Frobenious-Shur factorization

$$A-\mu=\begin{pmatrix}1 & G\\ 0 & 1\end{pmatrix}\begin{pmatrix}S-\mu & 0\\ 0 & A_{22}-\mu\end{pmatrix}\begin{pmatrix}1 & 0\\ F & 1\end{pmatrix}$$

where $G=G(\mu)$, $F=F(\mu)$ and $S=S(\mu)$ is the transfer function defined on the domain $\mathcal{D}(S)=\mathcal{D}_+$.

Lemma 1.

$$JA+\mu=J\begin{pmatrix}1 & G\\ 0 & 1\end{pmatrix}\begin{pmatrix}S+\mu & 0\\ 0 & A_{22}-\mu\end{pmatrix}\begin{pmatrix}1 & 0\\ F & 1\end{pmatrix}$$

Proof is obtained by direct verification.

Lemma 2. $\forall\mu\in\mathbb{C}^+$ *and* $\forall x\in\mathcal{D}_+$ *we have*

$$(Sx_+,x_+)=\left(JA\begin{pmatrix}x_+\\ -Fx_+\end{pmatrix},\begin{pmatrix}x_+\\ -Fx_+\end{pmatrix}\right)+\mu(Fx_+,Fx_+).$$

Proof is obtained by direct verification.

Corollary (important)**.** *$S=S(\mu)$ with domain $\mathcal{D}(S)=\mathcal{D}_+$ is dissipative in H_+ provided that assumption (i) holds. Also, S is closable. The closure of S is m-dissipative in H_+ $\Leftrightarrow$ A is m-dissipative in $\mathcal{K}$.*

Lemma 3 (important)**.** *Let a subspace $\mathcal{L}$ have a representation of the form*

$$\mathcal{L}=\{x\ :\ x=\begin{pmatrix}x_+\\ Kx_+\end{pmatrix},\ x_+\in H_+\}$$

where $K:H_+\to H_-$ is a bounded operator. Then $\mathcal{L}$ is A-invariant $\Leftrightarrow$

$$(1-KG)(A_{22}-\mu)(F+K)=K(S-\mu)$$

(the so-called Riccati equation for K).

Proof. For $x_+\in\mathcal{D}_+$

$$(A-\mu)\begin{pmatrix}x_+\\ Kx_+\end{pmatrix}=\begin{pmatrix}(S-\mu)x_++G(A_{22}-\mu)(F+K)x_+\\ (A_{22}-\mu)(F+K)x_+\end{pmatrix}.$$

Assuming that $\mathcal{L}$ is A-invariant we find $y_+\in H_+$ such that

$$[(S-\mu)+G(A_{22}-\mu)(F+K)]x_+=y_+,$$
$$(A_{22}-\mu)(F+K)x_+=Ky_+.$$

Substituting the first equality in the second one we come to Riccati equation for K.

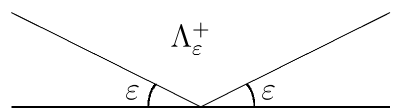
Conversely, Riccati equation for K implies the last two equations with some y_+, therefore the graph subspace $\mathcal{L}$ is A-invariant. □

Remark. Pontrjagin used the following version of Lemma 3: $\mathcal{L}$ is A-invariant $\Leftrightarrow$

$$A_{21}+A_{22}K-KA_{11}-KA_{12}K=0.$$

However this form of Riccati equation is inconvenient while working with unbounded entries A_{ij}.

Lemma 4. *Assume that $G(\mu)$ is compact for some $\mu \in \mathbb{C}^+$. Then it is compact for all $\mu \in \mathbb{C}^+$ and $\|G(\mu)\| \to 0$ as $\mu \to \infty$ and $\mu \in \Lambda_\varepsilon^+$.*



Proof is simple.

Lemma 5. *A subspace $\mathcal{L}$ is maximal nonnegative $\Leftrightarrow$ $\mathcal{L}$ has the graph representation*

$$\mathcal{L} = \{x = \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix}, \quad x_+ \in H_+\}$$

with the angle operator K, $\|K\| \leqslant 1$.

Corollary. *Take $\mu \in \mathbb{C}^+$ such that $\|G(\mu)\| < 1/2$. Then (a) holds $\Leftrightarrow$ $\exists$ there is a contraction K such that*

$$F + K = (A_{22} - \mu)^{-1}(1 - KG)^{-1}K(S - \mu).$$

Lemma 6. *Denote $H_S = \mathcal{D}(\overline{S}) \subset H_+$ where $\overline{S}$ is the closure of S and the norm in H_S is defined by*

$$\|x_+\|_{H_S} = \sqrt{\|\overline{S}x_+\|^2 + \|x_+\|^2}.$$

Then there is a complete orthogonal system $\{\varphi_k\}_1^\infty$ in H_+ such that $\{\varphi_k\}_1^\infty$ is a Riesz basis in H_S.

Proof. If H_S is compactly embedded in H_+ we take $\{\varphi_k\}_1^\infty$ consisting of eigenvectors of $S^*\overline{S}$. In general case additional work is required, however, this work is routine. ☐

Proof of Theorem A. Let P_n be orthogonal projectors onto $\mathrm{Lin}\{\varphi_k\}_1^n$ in H_+. Then $P_n \to 1$ in H_+ and $P_n \to 1$ in H_S.

Consider

$$A_n = \begin{pmatrix} P_n A_{11} P_n & P_n A_{12} \\ A_{21} P_n & A_{22} \end{pmatrix} \qquad \text{in } H_n^+ \oplus H^-, \ H_n^+ = P_n(H^+).$$

Then A_n is m-dissipative in Pontrjagin space Π_n and due to Krein–Langer–Azizov theorem (a) holds. This implies (Lemma 3) that

$$F_n + K_n = (A_{22} - \mu)^{-1}(1 - K_n G)^{-1} K_n (S_n - \mu). \tag{1}$$

It is known that the unit ball of a separable Hilbert space is weakly compact. The sequence of the operators K_n is bounded, hence, we can choose a weakly convergent subsequence $K_{n_j} \rightharpoonup K$. For short we omit the index j. Since $\|K_n\| \leqslant 1$, we have $\|K\| \leqslant 1$. Then

$$F_n = FP_n \to F, \quad K_n G \Rightarrow KG \text{ and } (1 - K_n G)^{-1} \Rightarrow 1 - KG$$

(we essentially use here that G is compact!).

Further,

$$K_n S_n = K_n S P_n,$$
$$\overline{S} P_n x \to \overline{S} x \qquad \forall x \in \mathcal{D}(\overline{S}),$$

Hence, $K_n S P_n x \rightharpoonup KSx$. Therefore we can pass to the weak limit in the equation (1) and obtain

$$F + K = (A_{22} - \mu)^{-1}(1 - KG)^{-1} K(S - \mu)$$

and by virtue of Lemma 3 property (a) holds. □

Let $A^+ = \overline{A}|_{\mathcal{L}^+}$. How to prove the property

(b) $\exists \mathcal{L}^+$ such that $\sigma(A^+) \subset \overline{\mathbb{C}^+}$?

We have

$$(\overline{A} - \mu)\begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix} = \begin{pmatrix} (\overline{S} - \mu + GL)x_+ \\ Lx_+ \end{pmatrix},$$

where $L := (A_{22} - \mu)(F + K)$, $\mathcal{D}(L) = \mathcal{D}(\overline{S})$.

Consider

$$Q : \mathcal{L} \to H_+ \quad \text{defined by } Q\begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix} = x_+.$$

Q is bounded and boundedly invertible, $\|Q^{-1}\| \leqslant 2$.

We have

$$\overline{A}|_{\mathcal{L}^+} = Q^{-1}(S + GL)Q = Q^{-1}[1 + G(1 - KG)^{-1}K(\overline{S} - \mu)]Q,$$

hence

$$(\overline{A} - \alpha)|_{\mathcal{L}^+} = Q^{-1}[1 + T(\alpha)](\overline{S}(\mu) - \alpha)Q, \tag{2}$$

where

$$T(\alpha) = G(1 - KG)^{-1}K(\overline{S} - \mu)(\overline{S} - \alpha)^{-1}$$

is a holomorphic operator function whose values are compact operators. Here we assumed that $(\overline{S} - \alpha)^{-1}$ exists $\Leftrightarrow$ $\overline{S}$ is m-dissipative in H_+ $\Leftrightarrow$ $\overline{A}$ is m-dissipative in $\mathcal{K}$. It can be shown that $\|T(\alpha)\| \to 0$ as $\alpha \to \infty$ along negative imaginary axis, therefore $1 + T(\alpha)$ has only discrete spectrum in $\mathbb{C}^-$.

Now we shall use the following

Lemma 7. $\operatorname{Im}[Ax_0, x_0] = (\operatorname{Im}\alpha_0)\,[x_0, x_0]$ *if* $Ax_0 = \alpha_0 x_0$.

Therefore, all eigenvectors of A corresponding to the eigenvalues from $\mathbb{C}^-$ are of negative type, provided that A is strictly dissipative in $\mathcal{K}$. Hence, these vectors do not belong to $\mathcal{L}^+$. This means that the eigenvalues from $\mathbb{C}^-$ do not belong to the spectrum of A^+, therefore $\sigma(A^+) \in \overline{\mathbb{C}}^+$.

This proves Theorem B if we assume in addition that A (or A^+) is strictly dissipative in $\mathcal{K}$.

If not, we consider

$$A_\varepsilon = A + i\varepsilon P_+, \qquad \varepsilon > 0.$$

Assertion (a) is valid for A_ε, and it does not have spectrum in $\mathbb{C}^-$, since

$$\operatorname{Im}\left[(A+i\varepsilon P_+)\begin{pmatrix} x_+ \\ Kx_+\end{pmatrix}, \begin{pmatrix} x_+ \\ Kx_+\end{pmatrix}\right] \geqslant \varepsilon(x_+,x_+).$$

Write Riccati equation for A_ε:

$$F+K_\varepsilon = (A_{22}-\mu)^{-1}(1-K_\varepsilon G)^{-1}K_\varepsilon(S+i\varepsilon-\mu).$$

Take $\varepsilon_n \to 0$ and $K_{\varepsilon_n} =: K_n \rightharpoonup K$.

We have

$$A_\varepsilon^+ = Q^{-1}[1+T_\varepsilon(\alpha)](S+i\varepsilon-\alpha)Q$$

and

$$T_\varepsilon(\alpha) = G(1-K_\varepsilon G)^{-1}K_\varepsilon(S+i\varepsilon-\mu)(S+i\varepsilon-\alpha)^{-1} \rightrightarrows T(\alpha).$$

Since $1+T_\varepsilon(\alpha)$ is a holomorphic operator function of Fredholm type in $\mathbb{C}_-$, and boundedly invertible $\forall\alpha\in\mathbb{C}^-$, so is $1+T(\alpha)$. $\Box$ (It is important here that $T_\varepsilon(\alpha)$ converges to $T(\alpha)$ uniformly, and $1+T(\alpha)$ has discrete spectrum in $\mathbb{C}^-$).

Theorems C–E are proved by analyzing representation (2).

For convenience we present here references related to the background of the problem.

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